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REALIZABILITY OF THE TORUS AND THE PROJECTIVE PLANE IN $\mathbb{R}^4$

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ABSTRACT

We show that every triangulation of the projective plane or the torus is isomorphic to a subcomplex of the boundary complex of a simplicial 5-dimensional convex polytope and thus linearly embeddable in $\mathbb{R}^4$.

In [1] it was proved that every triangulation of the projective plane is linearly embeddable in $\mathbb{R}^4$. Here we give a simpler proof of this observation and we show additionally that the same holds for the torus. We show in addition that they are convexly embeddable, i.e., every such triangulation is a subcomplex of the boundary complex of a simplicial 5-dimensional convex polytope.

We use an easy provable lemma being a modification of lemma 2 in [4].

LEMMA: *Let U, V be two orthogonal subspaces of $\mathbb{R}^n$ and $P \subseteq U$, $Q \subseteq V$ two convex polytopes such that $\mathbf{0} \in \operatorname{rel\,int} P$ and $\mathbf{0} \in \operatorname{rel\,int} Q$. Let $\mathcal{F}$ be the boundary complex of P. Then $|\mathcal{F}|$ and Q are in general position (i.e. they are joinable) and their join $|\mathcal{F}|Q$ is a convex polytope.*

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THEOREM: *Each triangulation of the projective plane or the torus is a subcomplex of the boundary complex of some simplicial 5-dimensional convex polytope.*

Proof: Let $\mathcal{K}$ be a simplicial complex such that $|\mathcal{K}|$ is the projective plane or the torus. Let $\mathfrak{c}$ be a cycle representing a nontrivial element of the first homology group of $\mathcal{K}$ over the field of integers modulo 2 such that $\mathfrak{c}$ has a minimal number of edges. $\mathfrak{c}$ as a Eulerian graph is the sum of circles and, because of the minimality property, $\mathfrak{c}$ itself is a circle.

Let $\mathcal{L}$ be the subcomplex of $\mathcal{K}$ spanned by the vertices of $\mathfrak{c}$ (that is, the set of simplexes of $\mathcal{K}$ with vertex set in $\mathfrak{c}$). Then $\mathcal{L}$ consists of exactly the vertices and edges of $\mathfrak{c}$: $\mathfrak{c}$ cannot have a diagonal in $\mathcal{K}$ because otherwise $\mathfrak{c}$ would be a sum, $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2$, of two circles $\mathfrak{c}_1, \mathfrak{c}_2 \neq \mathbf{0}$, each consisting of the diagonal and of a part of $\mathfrak{c}$, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ being smaller than $\mathfrak{c}$ and one of them not being homologously trivial. $\mathcal{L}$ cannot contain a triangle σ, otherwise $\mathfrak{c}$ would be the boundary of σ.

Let $\mathcal{M}$ be the subcomplex of $\mathcal{K}$ consisting of all simplexes of $\mathcal{K}$ in which the vertices of $\mathcal{L}$ do not appear. Then $|\mathcal{M}|$ is planar because, if we remove a regular neighborhood of $\mathcal{L}$ from $\mathcal{K}$, there will remain a disk or a cylinder (for the methods of proving this see, e.g., [2]).

$\mathcal{M}$ as a planar set can be extended to a triangulation $\mathcal{M}'$ of the sphere (see [3], p. 36). By the theorem of Steinitz [5], $\mathcal{M}'$ is the boundary complex of a 3-dimensional convex polytope P. $\mathcal{L}$ can be considered as the boundary complex of a 2-dimensional convex polytope Q. By our lemma we can now construct a 5-dimensional convex polytope R by joining P and the boundary of Q such that the join $\mathcal{L}\mathcal{M}'$ is the boundary complex of R.

Every simplex $\sigma \in \mathcal{K}$ is of the form $\sigma = \tau\nu$ where $\tau \in \mathcal{L}$ and $\nu \in \mathcal{M}$. It follows that $\mathcal{K} \subset \mathcal{L}\mathcal{M} \subset \mathcal{L}\mathcal{M}'$, showing the assertion. ∎

Remark: The method of the proof cannot be applied for the Klein bottle or for other surfaces of higher genus. Take, for example, two triangulations of the projective plane, remove a triangle from each and glue them along their boundaries in order to get a triangulation $\mathcal{K}$ of the Klein bottle. Let a, b, c denote the edges of the common boundary of the two Möbius strips. Subdivide $\mathcal{K}$ to get a triangulation $\mathcal{K}'$ such that a, b and c are not subdivided in $\mathcal{K}$ and such that, after the removal of all simplexes having a nonempty intersection with a, b, c, there will remain two Möbius strips. Then $\mathcal{K}'$ has the following obvious property: If $\mathfrak{c}$ is a circle such that a removal of $\mathfrak{c}$ results in a planar complex then

$\mathfrak{c}$ has one of the edges a, b, c as a diagonal and therefore $\mathcal{L}$ contains not only the edges and vertices of $\mathfrak{c}$, with the terminology used above.

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